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The Best Interpolating Approximation and Weighted Approximation

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In an earlier note [1], we showed that the best interpolating approximation to $f \in C[a, b]$ was, under certain conditions, the limit of a sequence of unconstrained best approximations using a sequence of discontinuous weight functions. We now show that the discontinuity of the weight functions is not essential by refining the argument in [1]. We also generalize to $C(X)$, (X, d) a metric space. The approximating family is not required to be linear. $C(X)$ represents the bounded, continuous real-valued functions on X.

Specifically, let $f \in C(X)$ satisfy a Lipschitz condition: $|f(x) - f(y)| \leqslant L_f d(x, y)$ for all $x, y \in X$ and some $L_f \geqslant 0$. Let G be a family of functions in $C(X)$, $f \notin G$, with the property that the intersection of G with every closed ball in $C(X)$ is compact. Also, assume that each element of G satisfies a Lipschitz condition: if $g \in G$, there is an $L_g \geqslant 0$ such that $|g(x) - g(y)| \leqslant L_g d(x, y)$ for all $x, y \in X$. Let $Z = \{z_1, z_2,..., z_k\}$ be a set of points in X. An element $g^* \in G$ is called a best (uniform) approximation to f, interpolating on Z, if $g^*(x) = f(x)$ for all $x \in Z$ and $\|g^* - f\| \leqslant \|g - f\|$ whenever $g \in G$ and $g(x) = f(x)$ for all $x \in Z$. $\|\cdot\|$ represents the uniform norm on X. Given $\omega > 0$ on X, we say g^* is a best approximation to f with weight function ω if $\|\omega(f - g^*)\| \leqslant \|\omega(f - g)\|$ for all $g \in G$.

THEOREM. *If there is a unique $g^* \in G$ that is a best approximation to f interpolating on Z, then there is a sequence of weight functions $\langle \omega_i \rangle$, continuous on X, such that $g_i \to g^*$ uniformly as $i \to \infty$, where each g_i is a best approximation to f with weight function ω_i. One such sequence of ω_i's can be defined by $\omega_i(x) = \max(\omega_i^1(x), \omega_i^2(x),..., \omega_i^k(x))$, $i = 1, 2,...$, where ω_i^j is defined for $j = 1, 2,..., k$ and all i by*

$$\omega_i^j(x) = \max\{1, 1 + i^{1/2} - i^{3/2} d(x, z_j)\}.$$

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Proof. It is readily verified that each ω_i is continuous. We first show that $\| g_i - f \| \to \| g^* - f \|$. Indeed $\| g_i - f \| \leqslant \| \omega_i(g_i - f) \| \leqslant \| \omega_i(g^* - f) \|$. Also $\| \omega_i(g^* - f) \| \leqslant \| g^* - f \|$ for sufficiently large i since for each j and x with $d(x, z_j) \leqslant 1/i$, and i sufficiently large $|\omega_i(x)(g^*(x) - f(x))| \leqslant [1 + i^{1/2}](L_f + L_{g^*})(1/i)$ which approaches zero as $i \to \infty$, whereas $\| \omega_i(g^* - f) \|$ is bounded away from zero. Therefore, g_i is bounded and has a subsequence, call it $\langle g_{i(m)} \rangle$ such that $g_{i(m)} \to \bar{g} \in G$, by the properties of G. Clearly $\| \bar{g} - f \| \leqslant \| g^* - f \|$. $\bar{g}$ interpolates f on Z since $\omega_i^j(z_j) \to \infty$ for each j. Therefore $\| \bar{g} - f \| = \| g^* - f \|$ since otherwise $\bar{g}$ would be a better inerpolating approximation than g^*. By uniqueness, $\bar{g} = g^*$ and $g_{i(m)} \to g^*$. In fact $g_i \to g^*$ since otherwise there would be a second subsequence $\langle g_{i(n)} \rangle$, $g_{i(n)} \to \hat{g}$, $\hat{g} \neq \bar{g}$. But repeating the argument used for $\bar{g}$, we would obtain $\hat{g} = g^* = \bar{g}$, a contradiction.

COROLLARY. *Using the above notation, let G be a continuously differentiable Tchebycheff space of (finite) dimension n, defined on $[a, b]$ and assume $k = \text{card } Z \leqslant n$. Then there is a unique best approximation g^* to f, $g^* \in G$, interpolating on Z. Furthermore, $g_i \to g^*$ uniformly.*

Proof. For Tchebycheff spaces of finite dimension, the existence and uniqueness of g^* is well known [2], provided $k \leqslant n$. Since G is continuously differentiable, it is Lipschitz. Since G is finite dimensional and closed, the intersection of G with any closed ball is compact.

Remarks. The Lipschitz requirement can actually be completely removed, but then the form of the weight functions is not so simple and consistent, but heavily dependent on g^*.

Note that the hypothesis of uniqueness of g^* in the Theorem allows us to deduce that the sequence $\langle g_i \rangle$ converges to g^*. This hypothesis can be removed and we can still deduce that $\langle g_i \rangle$ has a subsequence that converges to a best approximation.

Uniform approximation with a continuous weight function is well understood and Remes-like algorithms can in theory be used for computational purposes [3]. Thus one has a computational procedure alternative to that given in [2], where the Remes algorithm is modified to handle the interpolatory constraints. However, the unpleasant behaviour of ω_i for large i suggests that the former method might be numerically impractical.

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